On Sufficient and Necessary Conditions in Bounded CTL

Renyan Feng[1,2] , **Erman Acar**[2] , **Stefan Schlobach**[2] , **Yisong Wang**[1] , **Wanwei Liu** [3]

[1]Guizhou University, P. R. China

[2]Vrije Universiteit Amsterdam, Netherlands

[3]National University of Defense Technology, P. R. China

fengrenyan@gmail.com, {k.s.schlobach, erman.acar}@vu.nl, yswang@gzu.edu.cn, wwliu@nudt.edu.cn

Abstract

Computation Tree Logic (CTL) is one of the central formalisms in formal verification. As a specification language, it is used to express a property that the system at hand is expected to satisfy. From both the verification and the system design points of view, some information content of such property might become irrelevant for the system due to various reasons e.g., it might become obsolete by time, or perhaps infeasible due to practical difficulties. Then, the problem arises on how to subtract such piece of information without altering the relevant system behaviour or violating the existing specifications. Moreover, in such a scenario, two crucial notions are informative: the strongest necessary condition (SNC) and the weakest sufficient condition (WSC) of a given property.

To address such a scenario in a principled way, we introduce a forgetting-based approach in CTL and show that it can be used to compute SNC and WSC of a property under a given model. We study its theoretical properties and also show that our notion of forgetting satisfies existing essential postulates. Furthermore, we analyse the computational complexity of basic tasks, including various results for the relevant fragment CTL_{AF}.

1 Introduction

Consider a car-manufacturing company which produces two types of automobiles: a sedan car (basically a four-doored classical passenger car) and a sports car. No matter a sedan or a sports car, both production lines are subject to a single standard criterion which is indispensable: safety restrictions. This shared feature is also complemented several major differences aligned with these types in general. That is, a sedan car is produced with a small engine, while a sports car is produced with a large one. Moreover, due to its large amount of production, the sedan car is subject to some very restrictive low-carbon emission regulations, while a sports car is not. On the verge of shifting to an upcoming new engine technology, the company aims to adapt the sedan production to electrical engines. Such major shift in production also comes with one in regulations; electric sedans are not subject to low-carbon emission restrictions any more. In fact, due to the difference in its underlying technology, a sedan car drastically emits much less carbon, hence such standard is obsolete, and can be dropped. Yet dropping some restrictions in a large and complex production system in automative industry, without affecting the working system components or violating dependent specifications is a non-trivial task.

Similar scenarios may arise in many different domains such as business-process modelling, software development, concurrent systems and more (Baier and Katoen 2008). In general, some information content of such property might become irrelevant for the system due to various reasons e.g., it might become obsolete by time like in the above example, or perhaps becomes infeasible due to practical difficulties. Then, the problem arises on how to subtract such piece of information without altering the behaviour of the relevant system components or violating the existing specifications. Moreover, in such a scenario, two logical notions introduced by E. Dijkstra in (Dijkstra 1978) are very informative: the *strongest necessary condition* (SNC) and the *weakest sufficient condition* (WSC) of a given specification. These correspond to *most general consequence* and the *most specific abduction* of such specification, respectively.

To address such a scenario in a principled way, we employ a method based on formal verification.[1] In particular, we introduce a *forgetting*-based approach in Computation Tree Logic (CTL) (Clarke and Emerson 1981) a central formalism in formal verification, and show that it can be used to compute SNC and WSC, in the same spirit of (Lin 2001).

The scenario we mentioned concerning car-engine manufacturing can be easily represented as a small example by the following Kripke structure $\mathcal{M} = (S, R, L, s_0)$ in Figure 1 on $V = \{sl, sr, se, le, lc\}$ whose elements correspond to *select*, *safety restrictions*, *small engine*, *large engine* and *low-carbon emission requirements*, respectively. Moreover, s_0 is the initial state where we select either choose producing an engine for *sedan* which corresponds to the state s_1 or a *sports car* which corresponds to the state s_2. Since only a single-type of production is possible at a time, after each production state (s_1 or s_2), we turn back to the initial state (s_0) to start over.

The notions of SNC and WSC were considered in the scope of formal verification among others, in generating counterexamples (Dailler et al. 2018) and refinement of system (Woodcock and Morgan 1990). On the *forgetting* side, it was first formally defined in propositional and

[1] This is especially useful for abstracting away the domain-dependent problems, and focusing on conceptual ones.

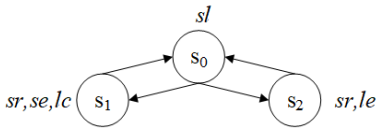

Figure 1: Car Engine Manufacturing Scenario

first order logics by Lin and Reiter (Lin and Reiter 1994). Over the last decades, researchers have developed forgetting notions and theories not only in classical logic but also in non-classical logic systems (Eiter and Kern-Isberner 2019), such as forgetting in logic programs under answer-set semantics (Zhang and Foo 2006; Eiter and Wang 2008; Wong 2009; Wang et al. 2012; Wang, Wang, and Zhang 2013), description logics (Wang et al. 2010; Lutz and Wolter 2011; Zhao and Schmidt 2017) and knowledge forgetting in modal logic (Zhang and Zhou 2009; Su et al. 2009; Liu and Wen 2011; Fang, Liu, and Van Ditmarsch 2019). It has also been considered in planning (Lin 2003) and conflict solving (Lang and Marquis 2010; Zhang, Foo, and Wang 2005), creating restricted views of ontologies (Zhao and Schmidt 2017), strongest and weakest definitions (Lang and Marquis 2008), SNC (WSC) (Lin 2001), among others.

Although forgetting has been extensively investigated from various aspects of different logical systems, the existing forgetting techniques are not directly applicable in CTL. For instance, in propositional forgetting theory, forgetting atom q from φ is equivalent to a formula $\varphi[q/\top] \vee \varphi[q/\ \bot]$, where $\varphi[q/X]$ is a formula obtained from φ by replacing each q with X ($X \in \{\top, \bot\}$). This method cannot be extended to a CTL formula. Consider a CTL formula $\psi = \text{AG}p \wedge \neg\text{AG}q \wedge \neg\text{AG}\neg q$. If we want to forget atom q from ψ by using the above method, we would have $\psi[q/\top] \vee \psi[q/\ \bot] \equiv \bot$. This is obviously not correct since after forgetting q this specification should not become inconsistent. Similar to (Zhang and Zhou 2009), we research forgetting in CTL from the semantic forgetting point of view. And it is shown that our definition of forgetting satisfies those four postulates of forgetting presented in (Zhang and Zhou 2009).

The rest of the paper is organised as follows. Section 2 introduces the notation and technical preliminaries. As key contributions, Section 3, introduces the notion of forgetting in bounded CTL, via developing the notion of V-bisimulation. Such bisimulation is constructed through a set-based bisimulation and more general than the classical bisimulation. Moreover, it provides a CTL characterization for model structures (with the initial state), and studies the semantic properties of forgetting. In addition, a complexity analysis, including a relevant fragment CTL_{AF}, is carried out. Section 4 explores the relation between forgetting and SNC (WSC). Section 5 gives a model-based algorithm for computing forgetting in CTL and outline its complexity. Conclusion closes the paper.

Due to space restrictions and to avoid hindering the flow of content, some of the proofs are moved to the supplementary material [2].

2 Preliminaries

We start with some technical and notational preliminaries. Throughout this paper, we fix a finite set $\mathcal{A}$ of propositional variables (or atoms), use V, V' for subsets of $\mathcal{A}$ and $\overline{V} = \mathcal{A} - V$.

2.1 Model structures in CTL

In general, a transition system can be described by a *model structure* (or *Kripke structure*) (see (Baier and Katoen 2008) for details). A model structure is a triple $\mathcal{M} = (S, R, L)$, where

- S is a finite nonempty set of states[3],
- $R \subseteq S \times S$ and, for each $s \in S$, there is $s' \in S$ such that $(s, s') \in R$,
- L is a labeling function $S \to 2^{\mathcal{A}}$.

Given a model structure $\mathcal{M} = (S, R, L)$, a *path* π_{s_i} starting from s_i of $\mathcal{M}$ is an infinite sequence of states $\pi_{s_i} = (s_i, s_{i+1}s_{i+2}, \dots)$, where for each j ($0 \leq i \leq j$), $(s_j, s_{j+1}) \in R$. By $s' \in \pi_{s_i}$ we mean that s' is a state in the path π_{s_i}. A state $s \in S$ is *initial* if for any state $s' \in S$, there is a path π_s s.t $s' \in \pi_s$. If s_0 is an initial state of $\mathcal{M}$, then we denote this model structure $\mathcal{M}$ as (S, R, L, s_0).

For a given model structure $\mathcal{M} = (S, R, L, s_0)$ and $s \in S$, the *computation tree* $\text{Tr}_n^{\mathcal{M}}(s)$ of $\mathcal{M}$ (or simply $\text{Tr}_n(s)$), that has depth n and is rooted at s, is recursively defined as (Browne, Clarke, and Grumberg 1988), for $n \geq 0$,

- $\text{Tr}_0(s)$ consists of a single node s with label s.
- $\text{Tr}_{n+1}(s)$ has as its root a node m with label s, and if $(s, s') \in R$ then the node m has a subtree $\text{Tr}_n(s')$.

A K-*structure* (or K-*interpretation*) is a model structure $\mathcal{M} = (S, R, L, s_0)$ associating with a state $s \in S$, which is written as $(\mathcal{M}, s)$ for convenience in the following. In the case $s = s_0$ is an initial state of $\mathcal{M}$, the K-structure is *initial*.

2.2 Syntax and semantics of CTL

In the following we briefly review the basic syntax and semantics of the CTL (Clarke, Emerson, and Sistla 1986). The *signature* of the language $\mathcal{L}$ of CTL includes:

- a finite set of Boolean variables, called *atoms* of $\mathcal{L}$: $\mathcal{A}$;
- constant symbols: $\bot$ and $\top$;
- the classical connectives: $\vee$ and $\neg$;
- the path quantifiers: A and E;
- the temporal operators: X, F, G U and W, that means 'neXt state', 'some Future state', 'all future states (Globally)', 'Until' and 'Unless', respectively;

[2] https://github.com/fengrenyan/proof-of-CTL.git

[3] Since CTL has finite model property (Emerson and Halpern 1985), we assume that the signature of states is fixed and finite, *i.e.* $S \subseteq \mathcal{S}$ with $\mathcal{S} = \{b_1, \dots, b_m\}$, such that any CTL formula with bounded length is satisfiable if and only if it is satisfiable in a such model structure. Thus, there are only finite number of model structures.

- parentheses: (and).

The *(existential normal form or ENF in short) formulas* of $\mathcal{L}$ are inductively defined via a Backus Naur form:

$$\phi ::= \bot \mid \top \mid p \mid \neg\phi \mid \phi \vee \phi \mid \text{EX}\phi \mid \text{EG}\phi \mid \text{E}[\phi \text{ U } \phi] \quad (1)$$

where $p \in \mathcal{A}$. The formulas $\phi \wedge \psi$ and $\phi \rightarrow \psi$ are defined in a standard manner of propositional logic. The other form formulas of $\mathcal{L}$ are abbreviated using the forms of (1). In the following we assume every formula of $\mathcal{L}$ has bounded size, where the size $|\varphi|$ of formula φ is its length over the alphbet of $\mathcal{L}$ (Emerson and Halpern 1985).

We are now in the position to recall the semantics of $\mathcal{L}$. Let $\mathcal{M} = (S, R, L, s_0)$ be a model structure, $s \in S$ and ϕ a formula of $\mathcal{L}$. The *satisfiability* relationship between $(\mathcal{M}, s)$ and ϕ, written $(\mathcal{M}, s) \models \phi$, is inductively defined on the structure of ϕ as follows:

- $(\mathcal{M}, s) \not\models \bot$ and $(\mathcal{M}, s) \models \top$;
- $(\mathcal{M}, s) \models p$ iff $p \in L(s)$;
- $(\mathcal{M}, s) \models \phi_1 \vee \phi_2$ iff $(\mathcal{M}, s) \models \phi_1$ or $(\mathcal{M}, s) \models \phi_2$;
- $(\mathcal{M}, s) \models \neg\phi$ iff $(\mathcal{M}, s) \not\models \phi$;
- $(\mathcal{M}, s) \models \text{EX}\phi$ iff $(\mathcal{M}, s_1) \models \phi$ for some $s_1 \in S$ and $(s, s_1) \in R$;
- $(\mathcal{M}, s) \models \text{EG}\phi$ iff $\mathcal{M}$ has a path $(s_1 = s, s_2, \ldots)$ such that $(\mathcal{M}, s_i) \models \phi$ for each $i \geq 1$;
- $(\mathcal{M}, s) \models \text{E}[\phi_1 \text{U} \phi_2]$ iff $\mathcal{M}$ has a path $(s_1 = s, s_2, \ldots)$ such that, for some $i \geq 1$, $(\mathcal{M}, s_i) \models \phi_2$ and $(\mathcal{M}, s_j) \models \phi_1$ for each $1 \leq j < i$.

Similar to the work in (Browne, Clarke, and Grumberg 1988; Bolotov 1999), only initial K-structures are considered to be candidate models in the following, unless otherwise noted. Formally, an initial K-structure $\mathcal{K}$ is a *model* of a formula (or set of formulas) ϕ whenever $\mathcal{K} \models \phi$ (or $\mathcal{K} \models \psi$ for each $\psi \in \varphi$). We denote *Mod*(ϕ) the set of models of ϕ. ϕ is *satisfiable* if *Mod*$(\phi) \neq \emptyset$. Given two formulas (or sets of formulas) ϕ_1 and ϕ_2, $\phi_1 \models \phi_2$ we mean *Mod*$(\phi_1) \subseteq$ *Mod*(ϕ_2). And by $\phi_1 \equiv \phi_2$, we mean $\phi_1 \models \phi_2$ and $\phi_2 \models \phi_1$. In this case, ϕ_1 is *equivalent* to ϕ_2. The set of atoms occurring in ϕ_1 is denoted by *Var*(ϕ_1). The formula ϕ_1 is *irrelevant to* the atoms in a set V (or simply V*-irrelevant*), written IR(ϕ_1, V), if there is a formula ψ with *Var*$(\psi) \cap V = \emptyset$ such that $\phi_1 \equiv \psi$. The V*-irrelevant* of a set of formulas can be defined similarly.

3 Forgetting in CTL

In this section, we present the notion of forgetting in CTL and report its properties. For convenience, in the following we denote $\mathcal{M} = (S, R, L, s_0)$, $\mathcal{M}' = (S', R', L', s'_0)$, $\mathcal{M}_i = (S_i, R_i, L_i, s_0^i)$ and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ with $s_i \in S_i$ and $i \in \mathbb{N}$.

3.1 Set-based bisimulation

In the following we present the notion of bisimulation on a given signature. It is (somehow) an extension of the classical bisimulation for CTL in (Baier and Katoen 2008).

Let $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ with $i \in \{1, 2\}$,

- $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_0$ if $L_1(s_1) - V = L_2(s_2) - V$;
- for $n \geq 0$, $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}$ if:
  - $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_0$,
  - for every $(s_1, s'_1) \in R_1$, there is a $(s_2, s'_2) \in R_2$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_n$, and
  - for every $(s_2, s'_2) \in R_2$, there is a $(s_1, s'_1) \in R_1$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_n$,

  where $\mathcal{K}'_i = (\mathcal{M}_i, s'_i)$ with $i \in \{1, 2\}$.

Now, we define the notion of V-bisimulation between K-structures:

Definition 1 (V-bisimulation)**.** *Let $V \subseteq \mathcal{A}$. Given two K-structures $\mathcal{K}_1$ and $\mathcal{K}_2$ are V-*bisimilar*, denoted $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ if and only if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$. Moreover, two paths $\pi_i = (s_{i,1}, s_{i,2}, \ldots)$ of $\mathcal{M}_i$ with $i \in \{1, 2\}$ are V-*bisimilar *if $\mathcal{K}_{1,j} \leftrightarrow_V \mathcal{K}_{2,j}$ for every $j \geq 1$ where $\mathcal{K}_{i,j} = (\mathcal{M}_i, s_{i,j})$.*

On the one hand, the above set-based bisimulation is an extension of the bisimulation-equivalence of Definition 7.1 in (Baier and Katoen 2008) in the sense that if $V = \mathcal{A}$ then our bisimulation is almost the same as the latter. On the other hand, the above set-based bisimulation notion is similar to the state equivalence in (Browne, Clarke, and Grumberg 1988). But it is different in the sense that ours is defined on K-structures, while it is defined on states in (Browne, Clarke, and Grumberg 1988). What's more, the set-based bisimulation notion is also different from the state-based bisimulation notion of Definition 7.7 in (Baier and Katoen 2008), which is defined for states of a given K-structure.

Example 1. *Let $\mathcal{K}_1$, $\mathcal{K}_2$ be two initial K-structures described in Figure 2. It is easy to check $\mathcal{K}_1 \leftrightarrow_{\{y\}} \mathcal{K}_2$.*

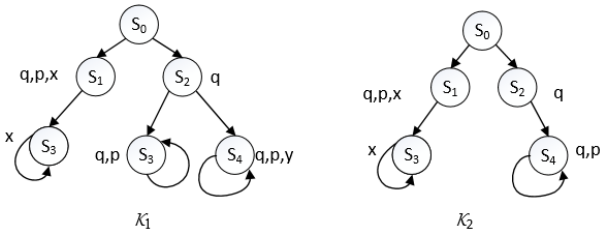

Figure 2: Two $\{y\}$-bisimular initial K-structures

It is apparent that $\leftrightarrow_V$ is a binary relation. In the sequel, we abbreviate $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ by $s_1 \leftrightarrow_V s_2$ whenever the underlying model structures of states s_1 and s_2 are clear from the context.

Lemma 1. *The relation $\leftrightarrow_V$ is an equivalence relation.*

Besides, we have the following properties:

Proposition 1. *Let $i \in \{1, 2\}$, $V_1, V_2 \subseteq \mathcal{A}$, s'_is be two states, π'_is be two paths, and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2, 3)$ be K-structures such that $\mathcal{K}_1 \leftrightarrow_{V_1} \mathcal{K}_2$ and $\mathcal{K}_2 \leftrightarrow_{V_2} \mathcal{K}_3$. Then:*

(i) $s'_1 \leftrightarrow_{V_i} s'_2$ $(i = 1, 2)$ implies $s'_1 \leftrightarrow_{V_1 \cup V_2} s'_2$;
(ii) $\pi'_1 \leftrightarrow_{V_i} \pi'_2$ $(i = 1, 2)$ implies $\pi'_1 \leftrightarrow_{V_1 \cup V_2} \pi'_2$;

(iii) for each path π_{s_1} of $\mathcal{M}_1$ there is a path π_{s_2} of $\mathcal{M}_2$ such that $\pi_{s_1} \leftrightarrow_{V_1} \pi_{s_2}$, and vice versa;

(iv) $\mathcal{K}_1 \leftrightarrow_{V_1 \cup V_2} \mathcal{K}_3$;

(v) If $V_1 \subseteq V_2$ then $\mathcal{K}_1 \leftrightarrow_{V_2} \mathcal{K}_2$.

Proof. We give proofs of (iii) and (iv) here. Proofs of other propositions can be found in the appendix. For convenience, we will refer to $\leftrightarrow_V$ by $\mathcal{B}$.

(iii) The following property show our result directly. Let $V \subseteq \mathcal{A}$ and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be K-structures. Then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ if and only if

(a) $L_1(s_1) - V = L_2(s_2) - V$,

(b) for every $(s_1, s_1') \in R_1$, there is $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$, and

(c) for every $(s_2, s_2') \in R_2$, there is $(s_1, s_1') \in R_1$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$,

where $\mathcal{K}_i' = (\mathcal{M}_i, s_i')$ with $i \in \{1, 2\}$.

We prove it from the following two aspects:

($\Rightarrow$) (a) It is apparent that $L_1(s_1) - V = L_2(s_2) - V$; (b) $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ iff $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$, then for each $(s_1, s_1') \in R_1$, there is a $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_{i-1}$ for all $i > 0$ and then $L_1(s_1') - V = L_2(s_2') - V$. Therefore, $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$. (c) This is similar with (b).

($\Leftarrow$) Apparently, $L_1(s_1) - V = L_2(s_2) - V$ implies that $(s_1, s_2) \in \mathcal{B}_0$; (b) implies that for every $(s_1, s_1') \in R_1$, there is $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_i$ for all $i \geq 0$; (c) implies that for every $(s_2, s_2') \in R_2$, there is $(s_1, s_1') \in R_1$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_i$ for all $i \geq 0$. Hence, we have $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$, and then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$.

(iv) Let $\mathcal{M}_i = (S_i, R_i, L_i, s_i)$ $(i = 1, 2, 3)$, $s_1 \leftrightarrow_{V_1} s_2$ via a binary relation $\mathcal{B}$, and $s_2 \leftrightarrow_{V_2} s_3$ via a binary relation $\mathcal{B}''$. Let $\mathcal{B}' = \{(w_1, w_3) | (w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}_2\}$. It's apparent that $(s_1, s_3) \in \mathcal{B}'$. We prove $\mathcal{B}'$ is a $V_1 \cup V_2$-bisimulation containing (s_1, s_3) from the (a), (b) and (c) of the previous step (iii) of X-bisimulation (where X is a set of atoms). For all $(w_1, w_3) \in \mathcal{B}'$:

(a) there is $w_2 \in S_2$ such that $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$, and $\forall q \notin V_1$, $q \in L_1(w_1)$ iff $q \in L_2(w_2)$ by $w_1 \leftrightarrow_{V_1} w_2$ and $\forall q' \notin V_2$, $q' \in L_2(w_2)$ iff $q' \in L_3(w_3)$ by $w_2 \leftrightarrow_{V_2} w_3$. Then we have $\forall r \notin V_1 \cup V_2$, $r \in L_1(w_1)$ iff $r \in L_3(w_3)$.

(b) if $(w_1, u_1) \in \mathcal{R}_1$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_1, u_2) \in \mathcal{B}$ (due to $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$ by the definition of $\mathcal{B}'$); and then $\exists u_3 \in S_3$ such that $(w_3, u_3) \in \mathcal{R}_3$ and $(u_2, u_3) \in \mathcal{B}''$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(c) if $(w_3, u_3) \in \mathcal{R}_3$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_2, u_3) \in \mathcal{B}_2$; and then $\exists u_1 \in S_1$ such that $(w_1, u_1) \in \mathcal{R}_1$ and $(u_1, u_2) \in \mathcal{B}$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

□

The (iv) in this proposition shows that if a K-structure is V_1 and V_2-bisimulation with the other two K-structures respectively, then those two K-structures are $V_1 \cup V_2$-bisimulation. This is important for forgetting. Besides, the (v) means that if two K-structures are V_1-bisimulation then they are V_2-bisimulation for each V_2 with $V_1 \subseteq V_2 \subseteq \mathcal{A}$.

Intuitively, if two K-structures are V-bisimilar, then they satisfy the same formula φ that does not contain any atoms in V, *i.e.* $\mathrm{IR}(\varphi, V)$.

Theorem 1. *Let $V \subseteq \mathcal{A}$, $\mathcal{K}_i$ $(i = 1, 2)$ be two K-structures such that $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ and ϕ a formula with IR(ϕ, V). Then $\mathcal{K}_1 \models \phi$ if and only if $\mathcal{K}_2 \models \phi$.*

Proof. (sketch) This can be proved by induction on the structures of ϕ. For instance, let $\phi = \psi_1 \vee \psi_2$, the induction hypothesis is $\mathcal{K}_1 \models \psi_i$ iff $\mathcal{K}_2 \models \psi_i$ with $i \in \{1, 2\}$. Then we can see that $\mathcal{K}_1 \models \phi$ iff $\mathcal{K}_1 \models \psi_1$ or $\mathcal{K}_1 \models \psi_2$ iff $\mathcal{K}_2 \models \psi_1$ or $\mathcal{K}_2 \models \psi_2$ by induction hypothesis. □

Example 2. *Let $\varphi_1 = \neg p \wedge \mathrm{AX}q \wedge \mathrm{EX}(x \rightarrow \mathrm{EX}x)$ and $\varphi_2 = q \wedge \mathrm{AX}q$ be two CTL formulae. They are $\{y\}$-irrelevant. One can check that $\mathcal{K}_1$ and $\mathcal{K}_2$ in Figure 2 satisfy φ_1, but they do not satisfy φ_2.*

Let $V \subseteq \mathcal{A}$, $\mathcal{M}_i$ $(i = 1, 2)$ be model structures. A computation tree $\mathrm{Tr}_n(s_1)$ of $\mathcal{M}_1$ is *V-bisimilar* to a computation tree $\mathrm{Tr}_n(s_2)$ of $\mathcal{M}_2$, written $(\mathcal{M}_1, \mathrm{Tr}_n(s_1)) \leftrightarrow_V (\mathcal{M}_2, \mathrm{Tr}_n(s_2))$ (or simply $\mathrm{Tr}_n(s_1) \leftrightarrow_V \mathrm{Tr}_n(s_2)$), if

- $L_1(s_1) - V = L_2(s_2) - V$,
- for every subtree $\mathrm{Tr}_{n-1}(s_1')$ of $\mathrm{Tr}_n(s_1)$, $\mathrm{Tr}_n(s_2)$ has a subtree $\mathrm{Tr}_{n-1}(s_2')$ such that $\mathrm{Tr}_{n-1}(s_1') \leftrightarrow_V \mathrm{Tr}_{n-1}(s_2')$, and
- for every subtree $\mathrm{Tr}_{n-1}(s_2')$ of $\mathrm{Tr}_n(s_2)$, $\mathrm{Tr}_n(s_1)$ has a subtree $\mathrm{Tr}_{n-1}(s_1')$ such that $\mathrm{Tr}_{n-1}(s_1') \leftrightarrow_V \mathrm{Tr}_{n-1}(s_2')$.

The last condition in the above definition holds trivially for $n = 0$.

Proposition 2. *Let $V \subseteq \mathcal{A}$ and $(\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be two K-structures. Then*

$(s_1, s_2) \in \mathcal{B}_n$ *iff* $\mathrm{Tr}_j(s_1) \leftrightarrow_V \mathrm{Tr}_j(s_2)$ *for every* $0 \leq j \leq n$.

Proof. (sketch) From the left to the right is apparent since $(s_1, s_2) \in \mathcal{B}_n$ implies that $(s_1, s_2) \in \mathcal{B}_m$ for every $0 \leq m \leq n$.

For the other direction, in order to show $(s_1, s_2) \in \mathcal{B}_n$ we need only to prove for any s_1' with $(s_1, s_1') \in R_1$ there is s_2' with $(s_2, s_2') \in R_2$ s.t. $(s_2, s_2') \in \mathcal{B}_{n-1}$ and vice versa. $\mathrm{Tr}_0(s_1) \leftrightarrow_V \mathrm{Tr}_0(s_2)$ implies $(s_1, s_2) \in B_0$, $\mathrm{Tr}_1(s_1) \leftrightarrow_V \mathrm{Tr}_1(s_2)$ implies for any s_1' with $(s_1, s_1') \in R_1$ there is s_2' with $(s_2, s_2') \in R_2$ s.t. $(s_2, s_2') \in B_0$ and vice versa, hence $(s_1, s_2) \in B_1$. Therefore, we can prove $(s_1, s_2) \in \mathcal{B}_n$ recursively. □

This means that $\mathrm{Tr}_j(s_1) \leftrightarrow_V \mathrm{Tr}_j(s_2)$ for all $j \geq 0$ if $s_1 \leftrightarrow_V s_2$, otherwise there is some k such that $\mathrm{Tr}_k(s_1)$ and $\mathrm{Tr}_k(s_2)$ are not V-bisimilar.

Proposition 3. *Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ be a model structure and $s, s' \in S$ such that $s \not\leftrightarrow_V s'$. There exists a least k such that $\mathrm{Tr}_k(s)$ and $\mathrm{Tr}_k(s')$ are not V-bisimilar.*

Proof. If $s \not\leftrightarrow_V s'$, then there exists a least constant c such that $(s_i, s_j) \notin \mathcal{B}_c$, and then there is a least constant m $(m \leq c)$ such that $\mathrm{Tr}_m(s_i)$ and $\mathrm{Tr}_m(s_j)$ are not V-bisimilar by Proposition 2. Let $k = m$, the lemma is proved. □

In this case, the model structure $\mathcal{M}$ is called V-*distinguishable* (by states s and s' at the least depth k), which is denoted by $\text{dis}_V(\mathcal{M}, s, s', k)$. The V-*characterization number* of $\mathcal{M}$, written $ch(\mathcal{M}, V)$, is defined as

$$ch(\mathcal{M}, V) = \begin{cases} \max\{k \mid s, s' \in S \text{ and } \text{dis}_V(\mathcal{M}, s, s', k)\}, \\ \qquad\qquad \mathcal{M} \text{ is } V\text{-distinguishable}; \\ \min\{k \mid \mathcal{B}_k = \mathcal{B}_{k+1}\}, \qquad \text{otherwise.} \end{cases}$$

3.2 Characterization of initial K-structure

In the following we present characterizing formulas of initial K-structures over a signature. As a basis, we first give the definition of characterizing formulas of computation trees.

Definition 2. *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ be a model structure and $s \in S$. The* characterizing formula *of the computation tree $Tr_n(s)$ on V, written $\mathcal{F}_V(Tr_n(s))$, is defined recursively as:*

$$\mathcal{F}_V(Tr_0(s)) = \bigwedge_{p \in V \cap L(s)} p \wedge \bigwedge_{q \in V - L(s)} \neg q,$$

$$\mathcal{F}_V(Tr_{k+1}(s)) = \bigwedge_{(s,s') \in R} \text{EX}\mathcal{F}_V(Tr_k(s')) \wedge \text{AX}\left(\bigvee_{(s,s') \in R} \mathcal{F}_V(Tr_k(s'))\right) \wedge \mathcal{F}_V(Tr_0(s))$$

for $k \geq 0$.

The characterizing formula of a computation tree formally exhibits the content of each node on V (i.e., atoms that are *true* at this node if they are in V, *false* otherwise) and the temporal relation between states recursively. The following result shows that the V-bisimulation between two computation trees implies the semantic equivalence of the corresponding characterizing formulas.

Lemma 2. *Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ and $\mathcal{M}'$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$. If $Tr_n(s) \leftrightarrow_{\overline{V}} Tr_n(s')$, then $\mathcal{F}_V(Tr_n(s)) \equiv \mathcal{F}_V(Tr_n(s'))$.*

Proof. (sketch) This result can be proved by induction on n.

For the base. It is apparent that for any $s \in S$ and $s' \in S'$, if $\text{Tr}_0(s) \leftrightarrow_{\overline{V}} \text{Tr}_0(s')$ then $\mathcal{F}_V(\text{Tr}_0(s)) \equiv \mathcal{F}_V(\text{Tr}_0(s'))$ due to $L(s) - \overline{V} = L'(s') - \overline{V}$ by known.

For the induction step. If $\text{Tr}_n(s) \leftrightarrow_{\overline{V}} \text{Tr}_n(s')$ we can prove for each state s_1 with $(s, s_1) \in R$ there is s_1' with $(s', s_1') \in R'$ such that $\mathcal{F}_V(\text{Tr}_n(s_1)) \equiv \mathcal{F}_V(\text{Tr}_n(s_1'))$ and versa vice. Then it is easy check $\mathcal{F}_V(\text{Tr}_n(s)) \equiv \mathcal{F}_V(\text{Tr}_n(s'))$. □

Let $s' = s$. It is clear that, for any formula φ of V, if φ is a characterizing formula of $\text{Tr}_n(s)$ then $\varphi \equiv \mathcal{F}_V(\text{Tr}_n(s))$.

Let $V \subseteq \mathcal{A}$, $\mathcal{K} = (\mathcal{M}, s_0)$ be an initial K-structure, $c = ch(\mathcal{M}, V)$ and $T(s') = \mathcal{F}_V(\text{Tr}_c(s'))$ for each state s' in $\mathcal{M}$. The *characterizing formula* of $\mathcal{K}$ on V, written $\mathcal{F}_V(\mathcal{M}, s_0)$ (or $\mathcal{F}_V(\mathcal{K})$ in short), is the following formula:

$$\mathcal{F}_V(\text{Tr}_c(s_0)) \wedge \bigwedge_{s \in S} \text{AG}\left(T(s) \rightarrow \bigwedge_{(s,s') \in R} \text{EX}T(s') \wedge \text{AX}(\bigvee_{(s,s') \in R} T(s'))\right)$$

It is apparent that $\text{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$. Besides, given a set of atomic propositions V, it is easy to check that any initial K-structure has its own characterizing formula on V. We will see later that the characterizing formula is critical to prove some important properties of forgetting and to compute the SNC and WSC of a property under an initial K-structure.

The following example illustrates how one can compute a characterizing formula:

Example 3. *Let $V = \{sr\}$ and $\overline{V} = \{sl, se, lc, le\}$ ($\mathcal{A} = \overline{V} \cup \{sr\}$), and $\mathcal{M}$ is as illustrated in Figure 1. We have $Tr_0(s_0) \not\leftrightarrow_{\overline{V}} Tr_0(s_1)$ and $Tr_0(s_0) \not\leftrightarrow_{\overline{V}} Tr_0(s_2)$, then $dis_{\overline{V}}(\mathcal{M}, s_0, s_1, 0)$ and $dis_{\overline{V}}(\mathcal{M}, s_0, s_2, 0)$. Besides, it is easy to check that $s_1 \leftrightarrow_{\overline{V}} s_2$ since they have the same direct successor s_0. Hence, $ch(\mathcal{M}, \overline{V}) = 0$. Therefore,*

$$\mathcal{F}_V(Tr_0(s_0)) = \neg sr$$
$$\mathcal{F}_V(Tr_0(s_1)) = \mathcal{F}_V(Tr_0(s_2)) = sr, \text{ and}$$
$$\mathcal{F}_V(\mathcal{M}, s_0) = \neg sr \wedge \text{AG}(\neg sr \rightarrow \text{AX}sr) \wedge \text{AG}(sr \rightarrow \text{AX}\neg sr).$$

The following theorem shows that the characterizing formulas of an initial K-structure are equivalent.

Theorem 2. *Given $V \subseteq \mathcal{A}$, let $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s_0')$ be two model structures. Then,*

(i) $(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathcal{M}, s_0)$ iff $(\mathcal{M}, s_0) \leftrightarrow_{\overline{V}} (\mathcal{M}', s_0')$;

(ii) $s_0 \leftrightarrow_{\overline{V}} s_0'$ implies $\mathcal{F}_V(\mathcal{M}, s_0) \equiv \mathcal{F}_V(\mathcal{M}', s_0')$.

Proof. (sketch) Let $k = ch(\mathcal{M}, V)$, On the one hand, one can verify that $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_n(s))$ holds for $n \geq 0$, which implies $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_k(s))$. On the other hand, $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_n(s'))$ implies $\text{Tr}_n(s) \leftrightarrow_{\overline{V}} \text{Tr}_n(s')$. In this case, it is not difficult to see that for each $s' \in S$, $(\mathcal{M}, s) \leftrightarrow_{\overline{V}} (\mathcal{M}, s')$ if and only if $(\mathcal{M}, s') \models \mathcal{F}_V(\text{Tr}_k(s))$.

(i) From the left to the right is apparent due to $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ and $\text{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$, hence $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$ by Theorem 1.

For the other direction. We can prove this by showing that $\forall n \geq 0$, $Tr_n(s_0) \leftrightarrow_{\overline{V}} Tr_n(s_0')$ by Proposition 2. The base case, *i.e.* $n = 0$, is easy. A key point for the induction step is that $(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathcal{M}, s_0)$ implies $\forall s' \in S'$, $(\mathcal{M}', s') \models \mathcal{F}_V(\text{Tr}_c(s)) \rightarrow \left(\bigwedge_{(s,s_1) \in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))\right) \wedge \text{AX}\left(\bigvee_{(s,s_1) \in R} \mathcal{F}_V(\text{Tr}_c(s_1))\right)$ for any $s \in S$. For further details, please see the supplementary material.

(ii) This is implied by Lemma 2. □

Recall that the number of model structures are finite. The next lemma is evident in terms of the above theorem.

Lemma 3. *Let φ be a formula. We have*

$$\varphi \equiv \bigvee_{(\mathcal{M}, s_0) \in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0). \tag{2}$$

It shows that any CTL formula can be equivalently transformed into a disjunction of the characterizing formulas for its models.

3.3 Semantic properties of forgetting in CTL

In this subsection we present the definition of forgetting in CTL and investigate its semantic properties.

Definition 3 (Forgetting)**.** *Let $V \subseteq \mathcal{A}$ and ϕ be a formula. A formula ψ with $Var(\psi) \cap V = \emptyset$ is a* result of forgetting V from ϕ*, if*

$$Mod(\psi) = \{\mathcal{K} \text{ is initial} \mid \exists \mathcal{K}' \in Mod(\phi) \ \&\ \mathcal{K}' \leftrightarrow_V \mathcal{K}\}.$$

Note that if both ψ and ψ' are results of forgetting V from ϕ, then $Mod(\psi) = Mod(\psi')$, i.e., ψ and ψ' have the same models. In this sense, the forgetting result is unique (up to equivalence). By Lemma 3, such a formula always exists, which is equivalent to

$$\bigvee_{\mathcal{K} \in \{\mathcal{K}' \mid \exists \mathcal{K}'' \in Mod(\phi) \text{ and } \mathcal{K}'' \leftrightarrow_V \mathcal{K}'\}} \mathcal{F}_{\overline{V}}(\mathcal{K}).$$

The forgetting result is denoted by $\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

Following from the knowledge forgetting point of view (Zhang and Zhou 2009), we show that the above forgetting repectes the four forgetting postulates in CTL:

- Weakening (**W**): $\varphi \models \varphi'$;
- Positive Persistence (**PP**): for any formula η, if $\mathrm{IR}(\eta, V)$ and $\varphi \models \eta$ then $\varphi' \models \eta$;
- Negative Persistence (**NP**): for any formual η, if $\mathrm{IR}(\eta, V)$ and $\varphi \not\models \eta$ then $\varphi' \not\models \eta$;
- Irrelevance (**IR**): $\mathrm{IR}(\varphi', V)$

where $V \subseteq \mathcal{A}$, φ is a formula and φ' is a result of forgetting V from φ. Intuitivly speaking, the postulate (**W**) says, forgetting weakens the original formula; the postulates (**PP**) and (**NP**) say that forgetting results have no effect on formulas that are irrelevant to fogotten atoms; the postulate (**IR**) states that forgetting result is irrelevant to forgotten atoms.

Theorem 3 (Representation Theorem)**.** *Let φ, φ' and ϕ be CTL formulas and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:*

(i) $\varphi' \equiv \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$,
(ii) $\varphi' \equiv \{\phi \mid \varphi \models \phi \text{ and } IR(\phi, V)\}$,
*(iii) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold.*

Proof. $(i) \Leftrightarrow (ii)$. To prove this, we will show that:

$$Mod(\mathrm{F}_{\mathrm{CTL}}(\varphi, V)) = Mod(\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\})$$
$$= Mod(\bigvee_{\mathcal{M}, s_0 \in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)).$$

Firstly, suppose that $(\mathcal{M}', s_0')$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$. Then there exists an initial K-structure $(\mathcal{M}, s_0)$ such that $(\mathcal{M}, s_0)$ is a model of φ and $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. By Theorem 1, we have $(\mathcal{M}', s_0') \models \phi$ for all ϕ that $\varphi \models \phi$ and $\mathrm{IR}(\phi, V)$. Thus, $(\mathcal{M}', s_0')$ is a model of $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\}$.

Secondly, suppose that $(\mathcal{M}', s_0')$ is a models of $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\}$. Thus, $(\mathcal{M}', s_0') \models \bigvee_{(\mathcal{M}, s_0) \in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)$ due to $\bigvee_{(\mathcal{M}, s_0) \in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)$ is irrelevant to V.

Finally, suppose that $(\mathcal{M}', s_0')$ is a model of $\bigvee_{\mathcal{M}, s_0 \in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)$. Then there exists $(\mathcal{M}, s_0) \in Mod(\varphi)$ such that $(\mathcal{M}', s_0') \models \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)$. Hence, $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$ by Theorem 2. Thus $(\mathcal{M}', s_0')$ is also a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$.

$(ii) \Rightarrow (iii)$. It is not difficult to prove it.

$(iii) \Rightarrow (ii)$. By Positive Persistence, we have $\varphi' \models \{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\}$. Now we show that $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\} \models \varphi'$. Otherwise, there exists formula ϕ' such that $\varphi' \models \phi'$ but $\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\} \not\models \phi'$. There are three cases:

- ϕ' is relevant to V. Thus, φ' is also relevant to V, a contradiction to Irrelevance.
- ϕ' is irrelevant to V and $\varphi \models \phi'$. This contradicts to our assumption.
- ϕ' is irrelevant to V and $\varphi \not\models \phi'$. By Negative Persistence, $\varphi' \not\models \phi'$, a contradiction.

Thus, φ' is equivalent to $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\}$. □

Given a formula $\varphi \wedge (q \leftrightarrow \alpha)$ with $q \notin Var(\varphi) \cup Var(\alpha)$. Let us show what this implies for forgetting.

Lemma 4. *Let φ and α be two CTL formulae and $q \in Var(\varphi) \cup Var(\alpha)$. Then $\mathrm{F}_{CTL}(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.*

The following proposition shows that the forgetting a set of atoms can be obtained by forgetting atoms in the set one by one.

Proposition 4. *Given a formula $\varphi \in$ CTL, V a set of atoms and p an atom such that $p \notin V$. Then,*

$$\mathrm{F}_{\mathrm{CTL}}(\varphi, \{p\} \cup V) \equiv \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi, p), V).$$

Proof. (stretch) Let $(\mathcal{M}_1, s_1)$ with $\mathcal{M}_1 = (S_1, R_1, L_1, s_1)$ be a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, \{p\} \cup V)$. By the definition of forgetting, there exists a model $(\mathcal{M}, s)$ with $\mathcal{M} = (S, R, L, s)$ of φ, such that $(\mathcal{M}_1, s_1) \leftrightarrow_{\{p\} \cup V} (\mathcal{M}, s)$. We can construct an initial K-structure $(\mathcal{M}_2, s_2)$ with $\mathcal{M}_2 = (S_2, R_2, L_2, s_2)$ such that $(\mathcal{M}, s) \leftrightarrow_{\{p\}} (\mathcal{M}_2, s_2)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$. Thus, we have $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\varphi, p)$ due to $\mathrm{IR}(\mathrm{F}_{\mathrm{CTL}}(\varphi, p), \{p\})$. Therefore, $(\mathcal{M}_1, s_1) \models \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi, p), V)$.

On the other hand, suppose that $(\mathcal{M}_1, s_1)$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi, p), V)$, then there exists an initial K-structure $(\mathcal{M}_2, s_2)$ such that $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\varphi, p)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$, and then there exists $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \varphi$ and $(\mathcal{M}, s) \leftrightarrow_{\{p\}} (\mathcal{M}_2, s_2)$. Therefore, $(\mathcal{M}, s) \leftrightarrow_{\{p\} \cup V} (\mathcal{M}_1, s_1)$ by Proposition 1, and consequently, $(\mathcal{M}_1, s_1) \models \mathrm{F}_{\mathrm{CTL}}(\varphi, \{p\} \cup V)$. □

The next corollary follows from the above proposition.

Corollary 4 (Commutativity)**.** *Let φ be a formula and $V_i \subseteq \mathcal{A}$ $(i = 1, 2)$. Then:*

$$\mathrm{F}_{\mathrm{CTL}}(\varphi, V_1 \cup V_2) \equiv \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi, V_1), V_2).$$

The following results, that hold in both classical propositional logic and modal logic **S5** (Zhang and Zhou 2009), further illustrate other important properties of CTL forgetting.

Proposition 5. *Let φ, φ_i, ψ_i ($i = 1, 2$) be formulas and $V \subseteq \mathcal{A}$. We have*

(i) $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ *is satisfiable iff* φ *is;*
(ii) If $\varphi_1 \equiv \varphi_2$, *then* $\mathrm{F}_{\mathrm{CTL}}(\varphi_1, V) \equiv \mathrm{F}_{\mathrm{CTL}}(\varphi_2, V)$;
(iii) If $\varphi_1 \models \varphi_2$, *then* $\mathrm{F}_{\mathrm{CTL}}(\varphi_1, V) \models \mathrm{F}_{\mathrm{CTL}}(\varphi_2, V)$;
(iv) $\mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2, V) \equiv \mathrm{F}_{\mathrm{CTL}}(\psi_1, V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2, V)$;
(v) $\mathrm{F}_{\mathrm{CTL}}(\psi_1 \wedge \psi_2, V) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1, V) \wedge \mathrm{F}_{\mathrm{CTL}}(\psi_2, V)$;

The next proposition shows that forgetting a set $V \subseteq \mathcal{A}$ from a formula with path quantifiers is equivalent to quantify the result of forgetting V from the formula with the same path quantifiers.

Proposition 6 (Homogeneity)**.** *Let $V \subseteq \mathcal{A}$ and $\phi \in$ CTL,*

(i) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V) \equiv \mathrm{AXF}_{\mathrm{CTL}}(\phi, V)$.
(ii) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V) \equiv \mathrm{EXF}_{\mathrm{CTL}}(\phi, V)$.
(iii) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AF}\phi, V) \equiv \mathrm{AFF}_{\mathrm{CTL}}(\phi, V)$.
(iv) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EF}\phi, V) \equiv \mathrm{EFF}_{\mathrm{CTL}}(\phi, V)$.

Proof. (stretch) We give the proof of (i), others can be proved similarly.

For one thing, for all model $\mathcal{K} = (\mathcal{M}, s_0)$ of the left side there is a model $\mathcal{K}' = (\mathcal{M}', s_0')$ of $\mathrm{AX}\phi$ such that $\mathcal{K} \leftrightarrow_V \mathcal{K}'$, i.e. $\forall s_1$ with $(s_0, s_1) \in R$ there is s_1' with $(s_0', s_1') \in R'$ such that $(\mathcal{M}, s_1) \leftrightarrow_V (\mathcal{M}', s_1')$ and then $(\mathcal{M}, s_1) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$ due to $\mathrm{IR}(\mathrm{F}_{\mathrm{CTL}}(\phi, V), V)$ and $(\mathcal{M}', s_1') \models \phi$ and versa vice. Therefore, $\mathcal{K} \models \mathrm{AXF}_{\mathrm{CTL}}(\phi, V)$.

For another, for all model $\mathcal{K} = (\mathcal{M}, s_0)$ of $\mathrm{AXF}_{\mathrm{CTL}}(\phi, V)$, we can easily construct an initial K-structure $\mathcal{K}' = (\mathcal{M}', s_0')$ such that $\mathcal{K} \leftrightarrow_V \mathcal{K}'$ and $\mathcal{K}' \models \mathrm{AX}\phi$ since for each model $\mathcal{K}_1$ of $\mathrm{F}_{\mathrm{CTL}}(\phi, V)$ there is a model $\mathcal{K}_2$ of ϕ s.t. $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$. Therefore, $\mathcal{K} \models \mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V)$ by the definition of forgetting. □

3.4 Complexity Results

In the following, we analyze the computational complexity of the various tasks regarding the forgetting in CTL and the fragment $\mathrm{CTL}_{\mathrm{AF}}$.

Proposition 7 (Model Checking on Forgetting)**.** *Let $(\mathcal{M}, s_0)$ be an initial* K*-structure, φ be a CTL formula and V a set of atoms. Deciding whether $(\mathcal{M}, s_0)$ is a model of $\mathrm{F}_{CTL}(\varphi, V)$ is NP-complete.*

Proof. One can (1) guess an initial K-structure $(\mathcal{M}', s_0')$ satisfying φ; and (2) check if $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. Both guessing and checking can be done in polynomial time. Hence, the problem is in NP. The hardness follows that the model checking for propositional variable forgetting is NP-hard (Zhang and Zhou 2008). □

The fragment $\mathrm{CTL}_{\mathrm{AF}}$ of CTL, in which each formula contains only AF temporal connective, corresponds to specifications that are desired to hold in all branches eventually. Such properties are of special interest in concurrent systems e.g., mutual exclusion and waiting events (Baier and Katoen 2008). In the following, we investigate some complexity results concerning forgetting and the logical entailment in this fragment.

Theorem 5 (Entailment on Forgetting)**.** *Let φ and ψ be two $\mathrm{CTL}_{\mathrm{AF}}$ formulas and V a set of atoms. Then, results:*

(i) deciding $\mathrm{F}_{CTL}(\varphi, V) \models^{?} \psi$ *is co-NP-complete,*
(ii) deciding $\psi \models^{?} \mathrm{F}_{CTL}(\varphi, V)$ *is* Π_2^P*-complete,*
(iii) deciding $\mathrm{F}_{CTL}(\varphi, V) \models^{?} \mathrm{F}_{CTL}(\psi, V)$ *is* Π_2^P*-complete.*

Proof. (i) It is known that deciding whether ψ is satisfiable is NP-Complete (Meier et al. 2015). The hardness is easy to see by setting $\mathrm{F}_{\mathrm{CTL}}(\varphi, \mathit{Var}(\varphi)) \equiv \top$, i.e., deciding whether ψ is valid. For membership, from Theorem 3, we have $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models \psi$ iff $\varphi \models \psi$ and $\mathrm{IR}(\psi, V)$. Clearly, in $\mathrm{CTL}_{\mathrm{AF}}$, deciding $\varphi \models \psi$ is in co-NP. We show that deciding whether $\mathrm{IR}(\psi, V)$ is also in co-NP. Without loss of generality, we assume that ψ is satisfiable. We consider the complement of the problem: deciding whether ψ is not irrelevant to V. It is easy to see that ψ is not irrelevant to V iff there exist a model $(\mathcal{M}, s_0)$ of ψ and an initial K-structure $(\mathcal{M}', s_0')$ such that $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$ and $(\mathcal{M}', s_0') \not\models \psi$. So checking whether ψ is not irrelevant to V can be achieved in the following steps: (1) guess two initial K-structures $(\mathcal{M}, s_0)$ and $(\mathcal{M}', s_0')$ such that $(\mathcal{M}, s_0) \models \psi$ and $(\mathcal{M}', s_0') \not\models \psi$, and (2) check $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. Obviously, both (1) and (2) can be done in polynomial time.

(ii) Membership. We consider the complement of the problem. We may guess an initial K-structure $(\mathcal{M}, s_0)$ satisfying ψ and check whether $(\mathcal{M}, s_0) \not\models \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$. From Proposition 7, we know that this is in Σ_2^P. So the original problem is in Π_2^P. Hardness. Let $\psi \equiv \top$. Then the problem is reduced to decide $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$'s validity. Since a propositional variable forgetting is a special case of temporal forgetting, the hardness is directly followed from the proof of Proposition 24 in (Lang, Liberatore, and Marquis 2003).

(iii) Membership. If $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \not\models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$ then there exist an initial K-structure $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ but $(\mathcal{M}, s) \not\models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$, i.e.,, there is $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ with $(\mathcal{M}_1, s_1) \models \varphi$ but $(\mathcal{M}_2, s_2) \not\models \psi$ for every $(\mathcal{M}_2, s_2)$ with $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$. It is evident that guessing such $(\mathcal{M}, s)$, $(\mathcal{M}_1, s_1)$ with $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ and checking $(\mathcal{M}_1, s_1) \models \varphi$ are feasible while checking $(\mathcal{M}_2, s_2) \not\models \psi$ for every $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$ can be done in polynomial time. Thus the problem is in Π_2^P.

Hardness. It follows from (ii) due to the fact that $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$ iff $\varphi \models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$ by $\mathrm{IR}(\mathrm{F}_{\mathrm{CTL}}(\psi, V), V)$.

□

The following corollary follow from Theorem 5.

Corollary 6. *Let φ and ψ be two $\mathrm{CTL}_{\mathrm{AF}}$ formulas and V a set of atoms. Then*

(i) deciding $\psi \equiv^{?} \mathrm{F}_{CTL}(\varphi, V)$ *is* Π_2^P*-complete,*
(ii) deciding $\mathrm{F}_{CTL}(\varphi, V) \equiv^{?} \varphi$ *is co-NP-complete,*
(iii) deciding $\mathrm{F}_{CTL}(\varphi, V) \equiv^{?} \mathrm{F}_{CTL}(\psi, V)$ *is* Π_2^P*-complete.*

4 Necessary and Sufficient Conditions

In this section, we present the definitions of strongest necessary and weakest sufficient conditions (SNC and WSC, respectively) of a specification in CTL and show that they can

be obtained by forgetting under a given initial K-structure and a set V of atoms.

Definition 4 (sufficient and necessary condition)**.** *Let ϕ be a formula (or an initial K-structure), ψ be a formula, $V \subseteq Var(\phi)$, $q \in Var(\phi) - V$ and $Var(\psi) \subseteq V$.*

- *ψ is a* necessary condition *(NC in short) of q on V under ϕ if $\phi \models q \to \psi$.*
- *ψ is a* sufficient condition *(SC in short) of q on V under ϕ if $\phi \models \psi \to q$.*
- *ψ is a* strongest necessary condition *(SNC in short) of q on V under ϕ if it is a NC of q on V under ϕ and $\phi \models \psi \to \psi'$ for any NC ψ' of q on V under ϕ.*
- *ψ is a* weakest sufficient condition *(WSC in short) of q on V under ϕ if it is a SC of q on V under ϕ and $\phi \models \psi' \to \psi$ for any SC ψ' of q on V under ϕ.*

Note that if both ψ and ψ' are SNC (WSC) of q on V under ϕ, then $Mod(\psi) = Mod(\psi')$, i.e., ψ and ψ' have the same models. In the sense of equivalence the SNC (WSC) is unique (up to equivalence).

Example 4. *Let $\psi = sl \to (\text{AX}sr \vee \text{EX}le \vee \text{EX}(se \vee lc))$, $\varphi = sl \to \text{AX}sr$, $\mathcal{A} = \{sl, sr, se, lc, le\}$ and $V = \{sl, sr\}$, then we can check that the WSC of ψ on V under the initial K-structure $(\mathcal{M}, s_0)$ in Figure 1 is φ.*

We verify this result by the following two steps:

(i) It is apparent that $\varphi \models \psi$ and $Var(\varphi) \subseteq V$. Besides, $(\mathcal{M}, s_0) \models \varphi \wedge \psi$, hence $(\mathcal{M}, s_0) \models \varphi \to \psi$, which means φ is a SC of ψ on V under $(\mathcal{M}, s_0)$,

(ii) We will show that for any SC φ' of ψ on V under $(\mathcal{M}, s_0)$, there is $(\mathcal{M}, s_0) \models \varphi' \to \varphi$. Following Figure 3, we can see that $(\mathcal{M}', s_0) \leftrightarrow_{\mathcal{A}\setminus V} (\mathcal{M}, s_0)$. By Theorem 2 we can easily obtain the characterizing formula of $(\mathcal{M}, s_0)$ on V, i.e. $\mathcal{F}_V(\mathcal{M}, s_0) \equiv \mathcal{F}_V(\mathcal{M}', s_0) = sl \wedge \neg sr \wedge \text{AG}((sl \wedge \neg sr) \to \text{AX}(\neg sl \wedge sr)) \wedge \text{AG}((\neg sl \wedge sr) \to \text{AX}(sl \wedge \neg sr))$, due to $ch(\mathcal{M}', \mathcal{A} \setminus V) = 0$.
If $\mathcal{F}_V(\mathcal{M}', s_0) \not\models \varphi'$ or $\varphi' \models \neg sl$ then we have $\mathcal{F}_V(\mathcal{M}', s_0) \models \varphi' \to \varphi$ i.e. $(\mathcal{M}, s_0) \models \varphi' \to \varphi$. Therefore, we suppose $\mathcal{F}_V(\mathcal{M}', s_0) \models \varphi'$. In this case we can construct φ' as follows: (1) sl is a sub-formula of φ' due to $\mathcal{F}_V(\mathcal{M}', s_0) \models \varphi'$; (2) $sl \to \text{AX}sr$ is also a sub-formula by (1), $(\mathcal{M}, s_0) \models \varphi' \to \psi$ and $Var(\varphi') \subseteq V$. This means that each SC φ' should be the following form with β is a CTL formula of V:

$$(sl \wedge (sl \to \text{AX}sr)) \wedge \beta.$$

In this case, we have $(\mathcal{M}, s_0) \models \varphi' \to \varphi$ for all SC φ' of ψ on V under $(\mathcal{M}, s_0)$ since for any α with $Var(\alpha) \subseteq V$ $(\mathcal{M}, s_0) \models \alpha$ iff $(\mathcal{M}', s_0) \models \alpha$ by Theorem 1.

Figure 3: A model structure $\mathcal{M}'$ with an initial state s_0

Proposition 8 (Dual)**.** *Let V, q, φ and ψ are the ones in Definition 4. The formula ψ is a SNC (WSC) of q on V under φ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on V under φ.*

Proof. (i) Suppose ψ is the SNC of q. Then $\varphi \models q \to \psi$. Thus $\varphi \models \neg\psi \to \neg q$. So $\neg\psi$ is a SC of $\neg q$. Suppose ψ' is any other SC of $\neg q$: $\varphi \models \psi' \to \neg q$. Then $\varphi \models q \to \neg\psi'$, this means $\neg\psi'$ is a NC of q on P under φ. Thus $\varphi \models \psi \to \neg\psi'$ by assumption. So $\varphi \models \psi' \to \neg\psi$. This proves that $\neg\psi$ is the WSC of $\neg q$. The proof of the other part of the proposition is similar.

(ii) The WSC case can be proved similarly with SNC case. □

This shows that the SNC and WSC are in fact dual conditions.

In order to generalise Definition 4 to arbitrary formulas, one can replace q (in the definition) by any formula α, and redefine V as a subset of $Var(\alpha) \cup Var(\phi)$. It turns out that the previous notion of SNC and WSC for an atomic proposition can be lifted to any formula, or conversely the SNC and WSC of any formula can be reduced to that of a proposition.

Proposition 9. *Let Γ and α be two formulas, $V \subseteq Var(\alpha) \cup Var(\Gamma)$ and q is a new proposition not in Γ and α. Then, a formula φ of V is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$.*

Proof. We prove this for SNC. The case for WSC is similar. Let $SNC(\varphi, \alpha, V, \Gamma)$ denote that φ is the SNC of α on V under Γ, and $NC(\varphi, \alpha, V, \Gamma)$ denote that φ is the NC of α on V under Γ.

($\Rightarrow$) We will show that if $SNC(\varphi, \alpha, V, \Gamma)$ holds, then $SNC(\varphi, q, V, \Gamma')$ will be true. According to $SNC(\varphi, \alpha, V, \Gamma)$ and $\alpha \equiv q$, we have $\Gamma' \models q \to \varphi$, which means φ is a NC of q on V under Γ'. Suppose φ' is any NC of q on V under Γ', then $\text{F}_{\text{CTL}}(\Gamma', q) \models \alpha \to \varphi'$ due to $\alpha \equiv q$, $IR(\alpha \to \varphi', \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \to \varphi'$ by Lemma 4, this means $NC(\varphi', \alpha, V, \Gamma)$. Therefore, $\Gamma \models \varphi \to \varphi'$ by the definition of SNC and $\Gamma' \models \varphi \to \varphi'$. Hence, $SNC(\varphi, q, V, \Gamma')$ holds.

($\Leftarrow$) We will show that if $SNC(\varphi, q, V, \Gamma')$ holds, then $SNC(\varphi, \alpha, V, \Gamma)$ will be true. According to $SNC(\varphi, q, V, \Gamma')$, it's not difficult to know that $\text{F}_{\text{CTL}}(\Gamma', \{q\}) \models \alpha \to \varphi$ due to $\alpha \equiv q$, $IR(\alpha \to \varphi, \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \to \varphi$ by Lemma 4, this means $NC(\varphi, \alpha, V, \Gamma)$. Suppose φ' is any NC of α on V under Γ. Then $\Gamma' \models q \to \varphi'$ since $\alpha \equiv q$ and $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$, which means $NC(\varphi', q, V, \Gamma')$. According to $SNC(\varphi, q, V, \Gamma')$, $IR(\varphi \to \varphi', \{q\})$ and (**PP**), we have $\text{F}_{\text{CTL}}(\Gamma', \{q\}) \models \varphi \to \varphi'$, and $\Gamma \models \varphi \to \varphi'$ by Lemma 4. Hence, $SNC(\varphi, \alpha, V, \Gamma)$ holds. □

The following result establishes the bridge between these two notions which are central to the paper.

Theorem 7. *Let φ be a formula, $V \subseteq Var(\varphi)$ and $q \in Var(\varphi) - V$.*

(i) $\text{F}_{\text{CTL}}(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$ is a SNC of q on V under φ.

(ii) $\neg\text{F}_{\text{CTL}}(\varphi \wedge \neg q, (Var(\varphi) \cup \{q\}) - V)$ is a WSC of q on V under φ.

Input: A CTL formula φ and a set V of atoms
Output: $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$
$\psi \leftarrow \bot$;
foreach *initial structure* $\mathcal{K}$ *(over* $\mathcal{A}$ *and* $\mathcal{S}$*)* **do**
  if $\mathcal{K} \not\models \varphi$ **then continue**;
  foreach *initial structure* $\mathcal{K}'$ *with* $\mathcal{K} \leftrightarrow_V \mathcal{K}'$ **do**
    $\psi \leftarrow \psi \vee \mathcal{F}_{\overline{V}}(\mathcal{K}')$;
  end
end
return ψ;

Algorithm 1: A model-based CTL forgetting procedure

Proof. We will prove the SNC part, while it is not difficult to prove the WSC part according to Proposition 8. Let $\mathcal{F} = \mathrm{F}_{\mathrm{CTL}}(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$.

The "NC" part: It's easy to see that $\varphi \wedge q \models \mathcal{F}$ by **(W)**. Hence, $\varphi \models q \rightarrow \mathcal{F}$, this means $\mathcal{F}$ is a NC of q on P under φ.

The "SNC" part: for all ψ', ψ' is the NC of q on V under φ, s.t. $\varphi \models \mathcal{F} \rightarrow \psi'$. Suppose that there is a NC ψ of q on V under φ and ψ is not logic equivalence with $\mathcal{F}$ under φ, s.t. $\varphi \models \psi \rightarrow \mathcal{F}$. We know that $\varphi \wedge q \models \psi$ iff $\mathcal{F} \models \psi$ by **(PP)**, since $IR(\psi, (Var(\varphi) \cup \{q\}) - V)$. Hence, $\varphi \wedge \mathcal{F} \models \psi$ by $\varphi \wedge q \models \psi$ (by suppose). We can see that $\varphi \wedge \psi \models \mathcal{F}$ by suppose. Therefore, $\varphi \models \psi \leftrightarrow \mathcal{F}$, which means ψ is logic equivalence with $\mathcal{F}$ under φ. This is contradict with the suppose. Then $\mathcal{F}$ is the SNC of q on P under φ. □

As aforementioned, since any initial K-structure can be characterized by a CTL formula, one can obtain the SNC (and its dual WSC) of a target property (a formula) under an initial K-structure by forgetting.

Theorem 8. *Let* $\mathcal{K} = (\mathcal{M}, s)$ *be an initial* K*-structure with* $\mathcal{M} = (S, R, L, s_0)$ *on the set* $\mathcal{A}$ *of atoms,* $V \subseteq \mathcal{A}$ *and* $q \in V' = \mathcal{A} - V$*. Then:*

(i) the SNC of q *on* V *under* $\mathcal{K}$ *is* $\mathrm{F}_{\mathrm{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge q, V')$.

(ii) the WSC of q *on* V *under* $\mathcal{K}$ *is* $\neg \mathrm{F}_{\mathrm{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge \neg q, V')$.

5 An Algorithm Computing CTL Forgetting

To compute the forgetting in CTL, we propose a model-based method. Intuitively, the model-based method means that we can compute the forgetting applied to a formula simply by considering all the possible models of that formula.

Next, we give a trivial algorithm computing CTL forgetting result, Algorithm 1. Its correctness is guaranteed by Lemma 3 and Theorem 2.

Example 5. *Consider the model given in Figure 1. Assume that we are given a property* $\alpha = \mathrm{AGEF}(lc \wedge sr)$ *and* $\{lc\}$*. Then, it is easy to check that* $\mathrm{F}_{\mathrm{CTL}}(\alpha, \{lc\}) \equiv \mathrm{AGEF}sr$.

Proposition 10. *Let* φ *be a CTL formula and* $V \subseteq \mathcal{A}$ *with* $|\mathcal{S}| = m$*,* $|\mathcal{A}| = n$ *and* $|V| = x$*. The the space complexity is* $O((n-x)m^{2(m+1)}2^{nm})$ *and the time complexity of Algorithm 1 is at least the same as the space.*

Proof. Supposing each state or atom occupy one byte, then a state pair (s, s') occupy two bytes. For any $B \subseteq \mathcal{S}$ with $B \neq \emptyset$ and $s_0 \in B$, we can construct an initial K-structure $(\mathcal{M}, s_0)$ with $\mathcal{M} = (B, R, L, s_0)$, in which there is at most $\frac{|B|^2}{2}$ state pairs in R and $|B| * n$ (s, A) $(A \subseteq \mathcal{A})$ in L. Hence, the $(\mathcal{M}, s_0)$ occupy at most $|B| + |B|^2 + |B| * n$ bytes. Besides, there is at most $|B|^{|B|+1} * 2^{nm}$ number of initial K-structures. Therefore, there is at most $m^{m+2} * 2^{nm}$ number of initial K-structures, hence it will at most cost $m^{m+2} * 2^{nm} * (m + m^2 + nm)$ bytes.

Let $k = n - x$, for any initial K-structure $\mathcal{K} = (\mathcal{M}, s_0)$ with $i \geq 1$ nodes, in the worst, i.e., $ch(\mathcal{M}, V) = i$, we will spend $N(i)$ space to store the characterizing formula. Where

$$\begin{aligned} N(i) &= (k + (\cdots + (k + 2ik) * (2i)) \cdots * (2i)) \\ &= (2i)^0 k + 2ik + (2i)^2 k + \cdots + (2i)^{(i-1)} k \\ &= \frac{(2i)^i - 1}{2i - 1} k. \end{aligned}$$

In the worst case, i.e., there is $m^{m+2} * 2^{nm}$ initial K-structures with m nodes, we will spent $m^{m+2} * 2^{nm} * N(m)$ bytes to store the result of forgetting.

Therefore, the space complexity is $O((n - x)m^{2(m+1)}2^{nm})$ and the time complexity is at least the same as the space. □

6 Concluding Remarks

Summary In this article, we generalized the notion of bisimulation from model structures of Computation Tree Logic (CTL) to model structures with initial states over a given signature V, named *V-bisimulation*. Based on this new bisimulation, we presented the notion of forgetting for CTL, which enables computing weakest sufficient and strongest necessary conditions of specifications. Further properties and complexity issues in this context were also explored. In particular, we have shown that our notion of forgetting satisfies the existing postulates of forgetting, which means it faithfully extends the notion of forgetting from classical propositional logic and modal logic S5 to CTL. On the complexity theory side, we investigated the model checking of forgetting, which turned out to be NP-complete, and the relevant fragment of $\mathrm{CTL}_{\mathrm{AF}}$ which ranged from co-NP to Π_2^p-completeness. And finally, we proposed a model-based algorithm which computes the forgetting of a given formula and a set of variables, and outlined its complexity.

Future work Note that, when a transition system $\mathcal{M}$ does not satisfy a specification ϕ, one can evaluate the weakest sufficient condition ψ over a signature V under which $\mathcal{M}$ satisfies ϕ, viz., $\mathcal{M} \models \psi \rightarrow \phi$ and ψ mentions only atoms from V. It is worthwhile to explore how the condition ψ can guide the design of a new transition system $\mathcal{M}'$ satisfying ψ.

Moreover, a further study regarding the computational complexity for other general fragments is required and part of the future research agenda. Such investigation can be coupled with fine-grained parameterized analysis.

References

Baier, C., and Katoen, J. 2008. *Principles of Model Checking*. The MIT Press.

Bolotov, A. 1999. A clausal resolution method for ctl branching-time temporal logic. *Journal of Experimental & Theoretical Artificial Intelligence* 11(1):77–93.

Browne, M. C.; Clarke, E. M.; and Grumberg, O. 1988. Characterizing finite kripke structures in propositional temporal logic. *Theor. Comput. Sci.* 59:115–131.

Clarke, E. M., and Emerson, E. A. 1981. Design and synthesis of synchronization skeletons using branching time temporal logic. In *Workshop on Logic of Programs*, 52–71. Springer.

Clarke, E. M.; Emerson, E. A.; and Sistla, A. P. 1986. Automatic verification of finite-state concurrent systems using temporal logic specifications. *ACM Trans. Program. Lang. Syst.* 8(2):244–263.

Dailler, S.; Hauzar, D.; Marché, C.; and Moy, Y. 2018. Instrumenting a weakest precondition calculus for counterexample generation. *Journal of logical and algebraic methods in programming* 99:97–113.

Dijkstra, E. W. 1978. Guarded commands, nondeterminacy, and formal derivation of programs. In *Programming Methodology*. Springer. 166–175.

Eiter, T., and Kern-Isberner, G. 2019. A brief survey on forgetting from a knowledge representation and reasoning perspective. *KI-Künstliche Intelligenz* 33(1):9–33.

Eiter, T., and Wang, K. 2008. *Semantic forgetting in answer set programming*. Elsevier Science Publishers Ltd.

Emerson, E. A., and Halpern, J. Y. 1985. Decision procedures and expressiveness in the temporal logic of branching time. *J. Comput. Syst. Sci.* 30(1):1–24.

Fang, L.; Liu, Y.; and Van Ditmarsch, H. 2019. Forgetting in multi-agent modal logics. *Artificial Intelligence* 266:51–80.

Lang, J., and Marquis, P. 2008. On propositional definability. *Artificial Intelligence* 172(8):991–1017.

Lang, J., and Marquis, P. 2010. *Reasoning under inconsistency: a forgetting-based approach*. Elsevier Science Publishers Ltd.

Lang, J.; Liberatore, P.; and Marquis, P. 2003. Propositional independence: Formula-variable independence and forgetting. *J. Artif. Intell. Res.* 18:391–443.

Lin, F., and Reiter, R. 1994. Forget it. In *Working Notes of AAAI Fall Symposium on Relevance*, 154–159.

Lin, F. 2001. On strongest necessary and weakest sufficient conditions. *Artif. Intell.* 128(1-2):143–159.

Lin, F. 2003. Compiling causal theories to successor state axioms and strips-like systems. *Journal of Artificial Intelligence Research* 19:279–314.

Liu, Y., and Wen, X. 2011. On the progression of knowledge in the situation calculus. In *IJCAI 2011, Proceedings of the 22nd International Joint Conference on Artificial Intelligence*, 976–982. Barcelona, Catalonia, Spain: IJCAI/AAAI.

Lutz, C., and Wolter, F. 2011. Foundations for uniform interpolation and forgetting in expressive description logics. In *IJCAI 2011, Proceedings of the 22nd International Joint Conference on Artificial Intelligence*, 989–995. Barcelona, Catalonia, Spain: IJCAI/AAAI.

Meier, A.; Thomas, M.; Vollmer, H.; and Mundhenk, M. 2015. Erratum: The complexity of satisfiability for fragments of CTL and ctl*. *Int. J. Found. Comput. Sci.* 26(8):1189.

Su, K.; Sattar, A.; Lv, G.; and Zhang, Y. 2009. Variable forgetting in reasoning about knowledge. *Journal of Artificial Intelligence Research* 35:677–716.

Wang, Z.; Wang, K.; Topor, R. W.; and Pan, J. Z. 2010. Forgetting for knowledge bases in DL-Lite. *Annuals of Mathematics and Artificial Intelligence* 58(1-2):117–151.

Wang, Y.; Zhang, Y.; Zhou, Y.; and Zhang, M. 2012. Forgetting in logic programs under strong equivalence. In *Principles of Knowledge Representation and Reasoning: Proceedings of the Thirteenth International Conference*, 643–647. Rome, Italy: AAAI Press.

Wang, Y.; Wang, K.; and Zhang, M. 2013. Forgetting for answer set programs revisited. In *IJCAI 2013, Proceedings of the 23rd International Joint Conference on Artificial Intelligence*, 1162–1168. Beijing, China: IJCAI/AAAI.

Wong, K.-S. 2009. *Forgetting in Logic Programs*. Ph.D. Dissertation, The University of New South Wales.

Woodcock, J. C., and Morgan, C. 1990. Refinement of state-based concurrent systems. In *International Symposium of VDM Europe*, 340–351. Springer.

Zhang, Y., and Foo, N. Y. 2006. Solving logic program conflict through strong and weak forgettings. *Artificial Intelligence* 170(8-9):739–778.

Zhang, Y., and Zhou, Y. 2008. Properties of knowledge forgetting. In Pagnucco, M., and Thielscher, M., eds., *Proceedings of the Twelfth International Workshop on Non-Monotonic Reasoning*, 68–75.

Zhang, Y., and Zhou, Y. 2009. Knowledge forgetting: Properties and applications. *Artificial Intelligence* 173(16-17):1525–1537.

Zhang, Y.; Foo, N. Y.; and Wang, K. 2005. Solving logic program conflict through strong and weak forgettings. In *Ijcai-05, Proceedings of the Nineteenth International Joint Conference on Artificial Intelligence, Edinburgh, Scotland, Uk, July 30-August*, 627–634.

Zhao, Y., and Schmidt, R. A. 2017. Role forgetting for alcoqh (δ)-ontologies using an ackermann-based approach. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence*, 1354–1361. AAAI Press.